RADON TRANSFORM ON REAL SYMMETRIC VARIETIES: KERNEL AND COKERNEL

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1. Introduction

Our concern is with

$$Y = G/H$$

a semisimple irreducible real symmetric variety (space).[1]

Our concern is with

$$L^2(Y)$$

the space of square integrable function on Y with respect to a G-invariant measure. This Hilbert space has a natural splitting

$$L^2(Y) = L^2_{\mathrm{mc}}(Y) \oplus L^2_{\mathrm{mc}}(Y)^\perp$$

into most continuous part and its orthocomplement.

Another function space is of need, namely:

$$\mathcal{A} := L^1(Y)^\omega\,,$$

the space of analytic vectors for the left regular representation of G on $L^1(Y)$. Further we set

$$\mathcal{A}_{\mathrm{mc}} := \mathcal{A} \cap L^2_{\mathrm{mc}}(Y) \quad \text{and} \quad \mathcal{A}_{\mathrm{mc}}^\perp := \mathcal{A} \cap L^2_{\mathrm{mc}}(Y)^\perp\,.$$

We believe that $\mathcal{A}_{\mathrm{mc}}$ is dense in $L^2(Y)_{\mathrm{mc}}$ and that $\mathcal{A}_{\mathrm{mc}}^\perp$ is dense in the subspace of $L^2(Y)_{\mathrm{mc}}^\perp$ which corresponds to principal series which are induced from integrable representations of their Levi subgroups – a proof of this and similar facts for $\mathcal{A}$ replaced by other function spaces is desirable.

Our concern is with an open domain in parameter space of generic real horospheres

$$\Xi = G/(M \cap H)N$$

where MAN is a minimal $\sigma\theta$-stable[2] parabolic subgroup of G.

Write $C_0^\omega(\Xi)$ for the space of analytic functions on Ξ which vanish at infinity. In this paper we verify the following facts:

- The map
$$\mathcal{R} : \mathcal{A} \to C_0^\omega(\Xi), f \mapsto \left(gM_HN \mapsto \int_N f(gnH)\;dn\right)$$
is well defined. (We call $\mathcal{R}$ the (minimal) Radon transform)
- $\mathcal{R}|_{\mathcal{A}_{\mathrm{mc}}^\perp} = 0$.
- $\mathcal{R}|_{\mathcal{A}_{\mathrm{mc}} \cap \mathcal{S}(Y)}$ is injective[3].

[1] This means G is a connected real semisimple Lie group, H is the fixed point group of an involutive automorphism σ of G such that there is no σ-stable normal subgroup $H \subset L \subset G$ with $\dim H < \dim L < \dim G$.

[2] θ is a Cartan involution commuting with σ.

[3] $\mathcal{S}(Y)$ is the Schwartz space of rapidly decaying functions

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2. **Real symmetric varieties**

2.1. **Notation**

The objective of this section is to introduce notation and to recall some facts regarding real symmetric varieties.

Let $G_\mathbb{C}$ be a simply connected linear algebraic group whose Lie algebra $\mathfrak{g}_\mathbb{C}$ we assume to be semi-simple. We fix a real form G of $G_\mathbb{C}$: this means that G is the fixed point set of an involutive automorphism σ of $G_\mathbb{C}$ and that $\mathfrak{g}$, the Lie algebra of G, yields $\mathfrak{g}_\mathbb{C}$ after complexifying.

Let now τ be a second involutive automorphism of $G_\mathbb{C}$ which we request to commute with σ. In particular, τ stabilizes G. We write

$$H_\mathbb{C} := G_\mathbb{C}^\tau \quad \text{and} \quad H := G^\tau$$

for the corresponding fixed point groups of τ in G, resp. $G_\mathbb{C}$. We note that $H_\mathbb{C}$ is always connected, but H usually is not; the basic example of $(G_\mathbb{C}, G) = (\mathrm{Sl}(2,\mathbb{C}), \mathrm{Sl}(2,\mathbb{R}))$ and $(H_\mathbb{C}, H) = (\mathrm{SO}(1,1;\mathbb{C}), \mathrm{SO}(1,1;\mathbb{R}))$ already illustrates the situation.

With G and H we form the object of our concern

$$Y = G/H\,;$$

we refer to Y as a *real (semi-simple) symmetric variety (or space)*. Henceforth we will denote by $y_o = H$ the standard base point in Y. We write $Y_\mathbb{C} = G_\mathbb{C}/H_\mathbb{C}$ for the affine complexification of Y and view, whenever convenient, Y as a subspace of $Y_\mathbb{C}$ via the embedding

$$Y \hookrightarrow Y_\mathbb{C}, \quad gH \mapsto gH_\mathbb{C}\,.$$

At this point it is useful to introduce infinitesimal notation. Lie groups will always be denoted by upper case Latin letters, e.g. G, H, K etc., and the corresponding Lie algebras by lower case German letters, eg. $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$ etc. It is convenient to use the same symbol τ for the derived automorphism $d\tau(\mathbf{1})$ of $\mathfrak{g}$. Let us denote by $\mathfrak{q}$ the -1-eigenspace

of τ on $\mathfrak{g}$. Note that $\mathfrak{q}$ is an H-module which naturally identifies with the tangent space $T_{y_o}Y$ at the base point.

From now we will request that Y is irreducible, i.e. we assume that the only τ-invariant ideals in $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$. In practice this means that G is simple except for the group case $G/H = H \times H/H \simeq H$.

We recall that maximal compact subgroups $K < G$ are in one-to-one correspondences with Cartan involutions $\theta : G \to G$. The correspondence is given by $K = G^\theta$. We form the Riemann symmetric space

$$X = G/K$$

of the non-compact type and denote by $x_o = K$ the standard base point. As before we write θ for the derived involution on $\mathfrak{g}$. We let $\mathfrak{p} \subset \mathfrak{g}$ be the -1-eigenspace θ and note that the K-module $\mathfrak{p}$ identifies with $T_{x_o}X$.

According to Berger, we may (and will) assume that K is τ-invariant. This implies that both $\mathfrak{h}$ and $\mathfrak{q}$ are τ-stable. Let us fix a maximal abelian subspace

$$\mathfrak{a} \subset \mathfrak{q} \cap \mathfrak{p}\,.$$

We wish to point out that $\mathfrak{a}$ is unique modulo conjugation by $H \cap K$, see [12], Lemma 7.1.5. Set $A = \exp(\mathfrak{a})$.

Our next concern is the centralizer $Z_G(A)$ of A. We first remark that $Z_G(A)$ is reductive and admits a natural splitting

$$Z_G(A) = A \times M\,,$$

(cf. [9], Prop. 7.82 (a)). The Lie algebra of M is given by

$$\mathfrak{m} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{a}^\perp$$

where $\mathfrak{a}^\perp$ is the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form κ of $\mathfrak{g}$. If M_0 denotes the connected component of M, then

$$M = M_0 F$$

where $F \subset M \cap K$ is a finite 2-group (cf. [9], Prop. 7.82 (d) and Th. 7.52).

Remark 2.1. *If $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, then $F \subset H$ as follows from the explicit description of F in* [9]*, Th. 7.52. In general however, the τ-stable group F is not contained in H and does not even admit a factorization $F = F^\tau F^{-\tau}$ in τ-fixed and τ-anti-fixed points.*

We write $\mathfrak{m}_{ns}$ for the non-compact semisimple part of $\mathfrak{m}$ and note that

$$\mathfrak{m}_{ns} \subset \mathfrak{h} \tag{2.1}$$

(cf. [12], Lemma 7.1.4). Set $M_H = M \cap H = Z_H(A)$ and let $\mathfrak{m} = \mathfrak{m}_h + \mathfrak{m}_q$ be the splitting of $\mathfrak{m}$ into $+1$ and -1-eigenspace. Note that $\mathfrak{m}_h$ is the Lie algebra of M_H. Then (2.1) implies that $\mathfrak{m}_q \subset \mathfrak{k}$ and consequently $M_q = \exp(\mathfrak{m}_q)$ is compact. Moreover:

$$M/F = M_H M_q/F \quad \text{with} \quad M_H \cap M_q \quad \text{discrete}\,.$$

We turn our attention to the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$. For $\alpha \in \mathfrak{a}^*$, let

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} \mid (\forall Y \in \mathfrak{a})\ [Y, X] = \alpha(Y)X\}$$

and set

$$\Sigma = \{\alpha \in \mathfrak{a}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq \{0\}\}\,.$$

It is a fact that Σ is a (possibly reduced) root system, cf. [12], Prop. 7.2.1. Hence we may fix a positive system $\Sigma^+ \subset \Sigma$ and define a corresponding nilpotent subalgebra

$$\mathfrak{n} := \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}^\alpha\,.$$

Set $N := \exp(\mathfrak{n})$. Note that $\tau(\mathfrak{n}) = \theta(\mathfrak{n})$. We record the decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \mathfrak{n} \oplus \tau(\mathfrak{n})\,.$$

We shift our focus to the real flag manifold of G associated to A and Σ^+. We define

$$P_{\min} := MAN$$

and note that $P_{\min}$ is a minimal $\theta\tau$-stable parabolic subgroup of G.

The open H-orbit decomposition on the flag manifold $G/P_{\min}$ is essential in the theory of H-spherical representations of G. In order to describe this decomposition we have to collect some facts on Weyl groups first.

Let us denote by $\mathcal{W}$ the Weyl group of the root system Σ. The Weyl group admits an analytic realization:

$$\mathcal{W} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})\,.$$

The group $\mathcal{W}$ features a natural subgroup

$$\mathcal{W}_H := N_{H\cap K}(\mathfrak{a})/Z_{H\cap K}(\mathfrak{a})\,.$$

Knowing $\mathcal{W}$ and $\mathcal{W}_H$, we can quote the decomposition of G into open $H \times P_{\min}$-cosets (cf. [11]):

$$G \doteq \amalg_{w\in\mathcal{W}_H\backslash\mathcal{W}} HwP_{\min}\,, \tag{2.2}$$

where $\doteq$ means equality up to a finite union of strictly lower dimensional $H\times P_{\min}$-orbits.

2.2. **Horospheres**

This paragraph is devoted to horospheres on the symmetric variety Y. By a *(generic) horosphere* on Y we understand an orbit of a conjugate of N of maximal dimension (i.e. $\dim N$). The entity of all horospheres will be denoted by $\mathrm{Hor}(Y)$. We remark that G acts naturally on $\mathrm{Hor}(Y)$ from the left.

Our goal is to show that $\mathrm{Hor}(Y)$ is a connected analytic manifold. For that we define

$$G_h := \{x\in G \mid Nx\cdot y_o \in \mathrm{Hor}(Y)\}$$

and note the following immediate things:

- G_h is open, right H-invariant and left $P_{\min}$-invariant.
- G_h contains the open $P_{\min}\times H$-cosets $P_{\min}wH$ where $w\in\mathcal{W}/\mathcal{W}_H$, see (2.2). In particular, G_h is dense.
- $\mathrm{Hor}(Y)=\{gNx\cdot y_o \mid g\in G, x\in G_h\}$.
- (Infinitesimal characterization) $G_h=\{x\in G\mid \mathrm{Ad}(x^{-1})\mathfrak{n}\cap\mathfrak{h}=\{0\}\}$.

Remark 2.2. *The set G_h is in general bigger then the open dense disjoint union $\bigcup_{w\in\mathcal{W}/\mathcal{W}_H} P_{\min}wH$. It is in particular connected as we show below. To see an example, consider $G=\mathrm{Sl}(2,\mathbb{R})$, $H=\mathrm{SO}(1,1;\mathbb{R})$ and $P_{\min}$ the upper triangular matrices with determinant one. Then $\mathfrak{h}$ and $\mathfrak{n}$, both one-dimensional, can never be conjugate. Thus, by the infinitesimal characterization from above, one has $G_h=G$ in this case.*

Next we provide charts for $\mathrm{Hor}(Y)$. For that we introduce the G-manifold

$$\Xi = G/M_HN$$

and define the G-equivariant map

$$E:\Xi\to\mathrm{Hor}(Y),\quad \xi=gM_HN\mapsto E(\xi)=gN\cdot y_o\,.$$

As in [8], Prop. 2.1, one verifies that E is an injection. Now we move our chart by elements $x\in G_h$. Set $L:=MA$, $L^x=x^{-1}Lx$, $L_H^x:=L^x\cap H$, $N^x:=x^{-1}Nx$ and $\Xi_x:=G/L_H^xN^x$. Then the map

$$E_x : G/L_H^x N^x \to \mathrm{Hor}(Y), \quad gL^x N^x \mapsto gN^x \cdot y_o \,.$$

is G-equivariant and injective. It is immediate that $(E_x, \Xi_x)_{x\in G_h}$ forms an analytic atlas for $\mathrm{Hor}(Y)$.

Lemma 2.3. $\mathrm{Hor}(Y)$ *is connected.*

Proof. It is sufficient to show that G_h is connected. We know that G_h is an open and dense $P_{\min} \times H$-invariant subset of G. Now there are only finitely many orbits of $P_{\min} \times H$ on G and those are described explicitly, see [11].

As G_h contains all open orbits, it is sufficient to show that G_h contains all codimension one orbits. This in turn follows from the explicit description of all orbits in [11], Th. 3 (i): if $HcP_{\min} \subset G$ is not open, then [11] implies that $\mathrm{Ad}(c)\mathfrak{a} \cap \mathfrak{h} \neq \{0\}$. In particular if $HcP_{\min}$ is of codimension one, then $\mathrm{Ad}(c)(\mathfrak{m}+\mathfrak{a}+\mathfrak{n})\cap\mathfrak{h} = \mathrm{Ad}(c)\mathfrak{a}\cap\mathfrak{h}$ and therefore $\mathrm{Ad}(c)\mathfrak{n} \cap \mathfrak{h} = \{0\}$. Our infinitesimal characterization completes the proof. $\square$

Finally we discuss polar coordinates on Ξ.

$$K/(M_H \cap K) \times A \to \Xi, \quad (k(M_H \cap K), a) \mapsto kaM_H N \tag{2.3}$$

is a diffeomorphism. Often we view A as subspace of Ξ via

$$A \hookrightarrow \Xi, \quad a \mapsto aM_H N \,.$$

3. Function spaces and the definition of the Radon transform

We consider the left regular representation L of G on $L^1(Y)$, i.e. for $g \in G$ and $f \in L^1(Y)$ we look at

$$[L(g)f](y) = f(g^{-1}y) \qquad (y \in Y)\,.$$

Then we focus on the subspace

$$\mathcal{A} := L^1(Y)^\omega$$

of analytic vectors for L. We note that $f \in \mathcal{A}$ means that $f \in C^\omega(Y)$ such that there exists an open neighborhood U of $\mathbf{1}$ in $G_\mathbb{C}$ such that f extends holomorphically to

$$UG \cdot y_o \subset Y_\mathbb{C} = G_\mathbb{C}/H_\mathbb{C}$$

(here we view Y embedded in $Y_\mathbb{C}$ via $gH \mapsto gH_\mathbb{C}$) such that for all compact $C \subset U$:

$$\sup_{c\in C} \|f(c\cdot)\|_{L^1(Y)} < \infty\,, \tag{3.1}$$

(cf. [6], Prop. A.2.1). For an open neighborhood U of $\mathbf{1}$ in $G_\mathbb{C}$ we denote by $\mathcal{A}_U$ the space of holomorphic functions on $UG\cdot y_o$ which satisfy (3.1) for all compact $C \subset U$. Note that $\mathcal{A}_U$ can be seen as a closed subspace of $\mathcal{O}(U, L^1(Y))$ and hence is a Fréchet space. Moreover

$$\mathcal{A} = \bigcup_U \mathcal{A}_U$$

with continuous inclusions

$$\mathcal{A}_U \to \mathcal{A}_V, \quad f \mapsto f|_{VG\cdot y_o}$$

for $V \subset U$. In this way we can endow $\mathcal{A}$ with a structure of of a locally convex space.

We observe that G, via L, acts on $\mathcal{A}$ in an analytic manner.

3.1. Definition of the Radon transform

We begin with the crucial technical fact.

Lemma 3.1. *Let $f \in \mathcal{A}$. Then the following assertions hold:*

(i) $\sup_{\substack{a\in A\\ k\in K}} \int_N |f(kan\cdot y_o)|\; dn < \infty$.

(ii) $\sup_{\substack{a\in A\\ k\in K}} a^{-2\rho} \int_N |f(kna\cdot y_o)|\; dn < \infty$.

Proof. (i) Let $f \in \mathcal{A}$. Let $B_\mathfrak{a} \subset \mathfrak{a}$ and $B_\mathfrak{n} \subset \mathfrak{n}$ be balls around zero and set

$$U_A := \exp(B_\mathfrak{a} + iB_\mathfrak{a}) \subset A_\mathbb{C}\,,$$
$$U_N := \exp(iB_\mathfrak{n})\exp(B_\mathfrak{n}) \subset N_\mathbb{C}\,.$$

If we choose $B_\mathfrak{a}$ and $B_\mathfrak{n}$ small enough, then f will extend to a holomorphic function in a neighborhood of

$$KU_NU_AG\cdot y_o$$

such that

$$\sup_{c\in KU_NU_A} \|f(c\cdot)\|_{L^1(Y)} < \infty\,.$$

To reduce notation let us assume that $M \subset H$ – this is no loss as the complementary piece M_qF in M to M_H is compact, (2.1). Then

$$NA \to Y, \quad na \mapsto na\cdot y_o$$

is an open embedding. In particular $NA\cdot y_o \subset Y$ is open. It follows that

$$U_A U_N AN \cdot y_o \subset Y_{\mathbb{C}}$$

is open and we may assume that

$$U_A U_N AN \subset A_{\mathbb{C}} N_{\mathbb{C}}$$

injects into $Y_{\mathbb{C}}$. Fix $k \in K$ and define a holomorphic function F on $U_N U_A AN$ by $F(z) := f(kz \cdot y_o)$. In particular, we obtain a constant $C > 0$ such that for all $an \in AN$ we get that

$$|F(an)| \leq C \int_{U_N U_A} |F(n'a'an)| da' dn'$$

with da' and dn' Haar-measures on $A_{\mathbb{C}}$ and $N_{\mathbb{C}}$ (Bergman-estimate). Let us write dy for a Haar measure on Y and observe that dy restricted to AN is just $da\ dn$ with da and dn Haar measures on A and N respectively. Therefore

$$\begin{aligned}
\int_N |F(an)| dn &\leq C \int_{U_N U_A} \int_N |F(n'a'an)|\ da'\ dn'\ dn \\
&\leq C \int_{U_N} \int_{B_{\mathfrak{a}}} \int_A \int_N |F(n' \exp(iX) a'an)|\ dn'\ dX\ da'\ dn \\
&\leq C \int_{U_N} \int_{B_{\mathfrak{a}}} \int_Y |f(kn' \exp(iX) y)|\ dn'\ dX\ dy \\
&\leq C \cdot \mathrm{vol}(U_N) \cdot \mathrm{vol}(B_{\mathfrak{a}}) \sup_{c \in K U_A U_N} \|f(c \cdot)\|_{L^1(Y)} < \infty\,.
\end{aligned}$$

We observe that the last expression does not depend on $a \in A$ and $k \in K$. This proves (i). Now (ii) is just a variable change of (i):

$$\begin{aligned}
\int_N |f(kna \cdot y_o)|\ dn &= \int_N |f(kaa^{-1}na \cdot y_o)|\ dn \\
&= a^{2\rho} \int_N |f(kan \cdot y_o)|\ dn
\end{aligned}$$

□

For $f \in \mathcal{A}$ we define a function $\mathcal{R}(f)$ on Ξ via

$$\mathcal{R}(f)(gM_H N) := \int_N f(gn \cdot y_o)\ dn\,.$$

According to our previous lemma the defining integrals are absolutely convergent.

Let us write $C_0^\omega(\Xi)$ for the space of analytic functions on Ξ which vanish at infinity. In view of (2.3) vanishing at infinity for F means

$$\lim_{\substack{a\to\infty\\ a\in A}} \sup_{k\in K} |F(kaM_H N)| = 0\,.$$

Proposition 3.2. *The following assertions hold:*

(i) *For all $f\in\mathcal{A}$ one has $\mathcal{R}(f)\in C_0^\omega(\Xi)$.*
(ii) *The map $\mathcal{R}:\mathcal{A}\to L^1(\Xi)^\omega\subset C_0^\omega(\Xi)$ is continuous.*

Proof. (i) As before it is no loss to assume that $M\subset H$ – the piece of M not in H is compact by (2.1). Let us first show that $F:=\mathcal{R}(f)|_A\in C_0^\omega$. In fact as f is in $L^1(Y)$ and $dy|_{AN}=da\ dn$, it follows that $F\in L^1(A)$. Moreover $F\in L^1(A)^\omega$, i.e. it is an analytic vector for the regular representation of A on $L^1(A)$. Therefore the standard Sobolev lemma implies that $F\in C_0^\omega(A)$.

Finally, employing the additional compact parameter $k\in K$ causes no difficulty.

(ii) This follows from (i) and the last (and crucial) estimate in the proof of Lemma 3.1(i). □

Remark 3.3. *Actually one can define the Radon transform with image on the whole horosphere space* $\mathrm{Hor}(Y)$. *Recall the set $G_h\subset G$ and for $x\in G_h$ the parameter space $\Xi_x=G/L_H^x N^x$. For $f\in\mathcal{A}$ one then defines*

$$\mathcal{R}_x(f)(gL_H^x N^x)=\int_N f(gx^{-1}nx\cdot y_o)\ dn\,.$$

The resulting function $\mathcal{R}_x(f)$ is then, as above seen to lie in $C_0^\omega(\Xi_x)\cap L^1(\Xi_x)$. Patching matters together we thus obtain a well defined G-map

$$\mathcal{R}:\mathcal{A}\to C^\omega(\mathrm{Hor}(Y))\,.$$

4. The kernel of the Radon transform: discrete spectrum

In this section we show that the discrete spectrum of $L^2(Y)$, as far as it meets $\mathcal{A}$, lies in the kernel of $\mathcal{R}$. In fact we show even more: namely that the trace of $\mathcal{A}$ in the orthocomplement of the most continuous spectrum lies in the kernel.

Recall our minimal $\theta\tau$-stable parabolic subgroup

$$P_{\min}=MAN\,.$$

In the sequel we use the symbol Q for a $\theta\tau$-stable parabolic which contains $P_{\min}$. There are only finitely many. We write

$$Q=M_QA_QN_Q$$

for its standard factorization and observe:

- $M_Q \supset M$,
- $A_Q \subset A$,
- $N_Q \subset N$.

One calls two parabolics Q and Q' associated if there exists an $n \in N_K(A)$ such that $nA_Q n^{-1} = A_{Q'}$. This induces an equivalence relation $\sim$ on our parabolics and we write $[Q]$ for the corresponding equivalence class. If the context is clear we simply omit the brackets and just write Q instead of $[Q]$.

It follows from the Plancherel theorem ([3], [5]) that

$$L^2(Y) = \bigoplus_{Q \supset P_{\min}/\sim} L^2(Y)_{[Q]}$$

where $L^2(Y)_Q = L^2(Y)_{[Q]}$ stands for the part corresponding to representations which are induced off from Q by discrete series of $M_Q/M_Q \cap wHw^{-1}$ with w running over representatives of $\mathcal{W}/\mathcal{W}_H$.

As $\mathcal{A} \subset C_0^\omega(Y)$, see [10], we observe that $\mathcal{A} \subset L^2(Y)$. However we note that $\mathcal{A}$ might not be dense in $L^2(Y)$: it has no components in this part of $L^2(Y)_Q$ which is induced from non-integrable discrete series of $M_Q/M_Q \cap wHw^{-1}$.

Let $\mathcal{A}_Q = L^2(Y)_Q \cap \mathcal{A}$. For the extreme choices of Q we use a special terminology:

$$L^2(Y)_{\text{disc}} := L^2(Y)_G \quad \text{and} \quad L^2(Y)_{\text{mc}} := L^2(Y)_{P_{\min}}$$

and one refers to the *discrete* and *most continuous* part of the square-integrable spectrum. Likewise we declare $\mathcal{A}_{\text{disc}}$ and $\mathcal{A}_{\text{mc}}$. Let us mention that we believe that

$$\mathcal{A}_{\text{mc}} \subset L^2(Y)_{\text{mc}} \quad \text{is dense}$$

(the heuristic reason for that is that $M/M \cap H$ is compact).

Theorem 4.1. $\mathcal{R}(\mathcal{A}_{\text{disc}}) = \{0\}$.

Proof. The proof is the same as for the group, see [13], Th. 7.2.2 for a useful exposition.

Let $f \in \mathcal{A}_{\text{disc}}$. We have to show that $\mathcal{R}(f) = 0$. As $\mathcal{R}$ is continuous (Proposition 3.2), standard density arguments reduce to the case where f belongs to a single discrete series representation and that f is K-finite. Let

$$V = \mathcal{U}(\mathfrak{g}_{\mathbb{C}})f$$

be the corresponding Harish-Chandra module and set $T := [\mathcal{R}|_V]|_A$. Then T factors over the Jacquet module $j(V) = V/\mathfrak{n}V$. We recall that

$j(V)$ is an admissible finitely generated $(M, \mathfrak{a})$-module. Hence

$$\dim \mathcal{U}(\mathfrak{a})T(f) < \infty .$$

Consequently

$$T(f)(a) = \sum_{\mu} a^{\mu} p_{\mu}(\log a) \qquad (a \in A)$$

where μ runs over a finite subset in $\mathfrak{a}_{\mathbb{C}}^*$ and p_μ is a polynomial (see [13], 8.A.2.10). From $T(f) \in C_0^\omega(A)$, we thus conclude that $T(f) = 0$ and hence $\mathcal{R}(f) = 0$ by the K-finiteness of f. □

As a consequence of the previous theorem we obtain the main result of this subsection.

Theorem 4.2. *Let $Q \supsetneq P_{\min}$. Then $\mathcal{R}(\mathcal{A}_Q) = \{0\}$.*

Proof. If $Q = G$, then this part of the previous theorem. The general case will be reduced to that. So suppose that $P_{\min} \subsetneq Q \subsetneq G$. Define $\Xi_Q = G/(M_Q \cap H)N_Q$ and like in (2.3) one has a diffeomorphic parameterization $[K \times_{M_Q \cap K} M_Q/M_Q \cap H] \times A \to \Xi_Q$.

As in Subsection 3.1, one obtains that the map

$$\mathcal{R}_Q : \mathcal{A} \to L^1(\Xi_Q)^\omega, \quad f \mapsto \left(g(M_Q \cap H)N_Q \mapsto \int_{N_Q} f(gnH)\ dn \right)$$

is defined, G-equivariant and continuous.

Next observe that

$$N = N_Q \rtimes N^Q$$

with $\{\mathbf{1}\} \neq N^Q \subset M_Q$. As before the map

$$\mathcal{R}^Q : L^1(\Xi_Q)^\omega \to L^1(\Xi)^\omega$$

$$f \mapsto \left(gM_H N \mapsto \int_{N^Q} f(gn(M_Q \cap H)N_Q)\ dn \right)$$

is defined, equivariant and continuous.

Now we note that

$$\mathcal{R} = \mathcal{R}^Q \circ \mathcal{R}_Q . \tag{4.1}$$

Let now $f \in \mathcal{A}_Q$. Without loss of generality we may assume that f belongs to a wave packet induced from a discrete series $\sigma \subset L^2(M_Q/M_Q \cap H)$.

Note that $M_Q/M_Q \cap H$ naturally embeds into Ξ_Q and that the restriction of $L^1(\Xi_Q)^\omega$ to $M_Q/M_Q \cap H$ stays integrable. Hence $F := \mathcal{R}_Q(f)$ restricted to $M_Q/M_Q \cap H$ is integrable as well.

We claim that $F|_{M_Q/M_Q\cap H}$ belongs to the σ-isotypical class. First note that $L^1(\Xi_Q)^\omega \subset L^2(\Xi_Q)$. By induction on stages

$$L^2(\Xi_Q) = \mathrm{Ind}_{(M_Q\cap H)N_Q}^G \mathrm{triv} \simeq \mathrm{Ind}_{M_Q N_Q}^G L^2(M_Q/M_Q\cap H)\,.$$

Thus if $L^2(M_Q/M_Q\cap H) = \int_{\widehat{M}_Q}^\oplus m_\pi \mathcal{H}_\pi \, d\mu(\pi)$ is the Plancherel decomposition, then as G-modules:

$$L^2(\Xi_Q) \simeq \int_{\hat{M}_Q}^\oplus m_\pi \,\mathrm{Ind}_{M_Q N_Q}^G \mathcal{H}_\pi \, d\mu(\pi)\,.$$

Now note that A_Q acts one right on Ξ_Q and this action commutes with G (see the next section for a detailed discussion for $Q = P_{\min}$). This gives us a further disintegration of the left regular representation L_Q of G on $L^2(\Xi_Q)$:

$$L_Q \simeq \int_{\widehat{M}_Q}^\oplus m_\pi \int_{i\mathfrak{a}_Q^*}^\oplus \mathrm{Ind}_{M_Q A_Q N_Q}^G [\pi\otimes(-\lambda-\rho_Q)\otimes \mathbf{1}]\; d\lambda \; d\mu(\pi)\,.$$

As R_Q is G-equivariant, we thus conclude that $R_Q(f) \in \mathrm{Ind}_{M_Q N_Q}^G \sigma$.

Our claim combined with the previous theorem implies that

$$\mathcal{R}^Q(F)|_{M_Q/M_H N^Q} = 0\,.$$

By the equivariance properties of $\mathcal{R}_Q$ and $\mathcal{R}^Q$ we are free to replace f (and hence F) by any G-translate. Consequently $\mathcal{R}^Q(F) = 0$, as was to be shown. □

5. Restriction of the Radon transform to the most continuous spectrum

The objective of this section is to show that $\mathcal{R}$ is faithful on the most continuous spectrum.

We recall a few facts on the spectrum of $L^2(\Xi)$ and the most continuous spectrum on Y and start with the "horocyclic picture". The homogeneous space Ξ carries a G-invariant measure. Consequently left shifts by G in the argument of a function on Ξ yields a unitary representation, say L, of G on $L^2(\Xi)$; in symbols

$$(L(g)f)(\xi) = f(g^{-1}\cdot\xi) \qquad (f\in L^2(\Xi), g\in G, \xi\in\Xi)\,.$$

It is important to note that the G-action on Ξ admits a commutating action of A from the right

$$\xi\cdot a = gaMN \qquad (\xi = gM_H N \in \Xi, a\in A);$$

this is because A normalizes $M_H N$. Therefore the description

$$(R(a)f)(\xi) = a^\rho \cdot f(\xi \cdot a) \qquad (f \in L^2(\Xi), a \in A, \xi \in \Xi)$$

defines a unitary representation $(R, L^2(\Xi))$ of A which commutes with the G-representation L. Accordingly we define an A-Fourier transform for an appropriate function f on Ξ by

$$\mathcal{F}_A(f)(\lambda, gM_H N) := \int_A [R(a)f](gM_H N) a^\lambda \; da \qquad (\lambda \in i\mathfrak{a}^*) \, .$$

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ let us set

$$L^2(\Xi)_\lambda := \{ f : G \to \mathbb{C} \mid \begin{array}{l} \bullet \ f \text{ measurable}, \\ \bullet \ f(\cdot man) = a^{-\rho-\lambda} f(\cdot) \ \forall man \in M_H AN, \\ \bullet \ \displaystyle\int_K |f(k)|^2 \; dk < \infty \} \end{array}$$

Likewise we write $C^\infty(\Xi)_\lambda$ for the smooth elements of $L^2(\Xi)_\lambda$. The disintegration of $L^2(\Xi)$ is then given by

$$L^2(\Xi) \simeq \int_{i\mathfrak{a}^*}^\oplus L^2(\Xi)_\lambda \; d\lambda$$

with isomorphism given by the A-Fourier transform

$$f \mapsto (\lambda \mapsto \mathcal{F}_A(f)(\lambda, \cdot)) \, .$$

In the next step we recall the Plancherel decomposition for the most continuous spectrum (cf. [1]).

Some generalities upfront. For a representation π of a group L on some topological vector space V we denote by π^* the dual representation on the (strong) topological dual V^* of V.

Let $\sigma \in \widehat{M/M_H}$ and V_σ a unitary representation module for σ.

For $\lambda \in \mathfrak{a}_\mathbb{C}^*$ we define

$$\mathcal{H}_{\sigma,\lambda} := \{ f : G \to V_\sigma \mid \begin{array}{l} \bullet \ f \text{ measurable}, \\ \bullet \ f(\cdot man) = a^{-\rho-\lambda} \sigma(m)^{-1} f(\cdot) \ \forall man \in P_{\min}, \\ \bullet \ \displaystyle\int_K \langle f(k), f(k) \rangle_\sigma \; dk < \infty \} \, . \end{array}$$

The group G acts on $\mathcal{H}_{\sigma,\lambda}$ by displacements from the left and the so-obtained Hilbert representation will be denoted by $\pi_{\sigma,\lambda}$.

Remark 5.1. *The relationship between $\mathcal{H}_{\sigma,\lambda}$ and $L^2(\Xi)_\lambda$ is as follows. If μ_σ is an (up to scalar unique) M_H-fixed element in V_σ^*, then the mapping*

$$\mathcal{H}_{\sigma,\lambda} \to L^2(\Xi)_\lambda, \quad f \mapsto \mu_\sigma(f)$$

is a G-equivariant injection. The map can be made isometric by appropriate scaling of μ_σ. Employing induction in stages one therefore obtains an isometric identification

$$\widehat{\bigoplus}_{\sigma \in \widehat{M/M_H}} \mathcal{H}_{\sigma,\lambda} = L^2(\Xi)_\lambda \,.$$

Sometimes it is useful to realize $\mathcal{H}_{\sigma,\lambda}$ as V_σ-valued functions on $\overline{N} := \theta(N)$; we speak of the non-compact realization then. Define a weight function on $\overline{N}$ by

$$w_\lambda(\overline{n}) = a^{2\,\mathrm{Re}\,\lambda}$$

where $a \in A$ is determined by $\overline{n} \in KaN$. Then the map

$$\mathcal{H}_{\sigma,\lambda} \to L^2(\overline{N}, w_\lambda(\overline{n})d\overline{n}) \otimes V_\sigma, \quad f \mapsto f|_{\overline{N}}$$

is an isometric isomorphism.

We remark that:

- $\pi_{\sigma,\lambda}$ is irreducible for generic λ.
- $\pi_{\sigma,\lambda}$ is unitary for $\lambda \in i\mathfrak{a}^*$.
- The dual representation of $\pi_{\sigma,\lambda}$ is canonically isomorphic to $\pi_{\sigma^*,-\lambda}$; the dual pairing is given by

$$\langle f, g\rangle := \int_{\overline{N}} (f(\overline{n}), g(\overline{n}))_\sigma \; d\overline{n}$$

for $f \in \mathcal{H}_{\sigma,\lambda}$, $g \in \mathcal{H}_{\sigma^*,-\lambda}$ and $(\,,\,)_\sigma$ the natural pairing between V_σ and V_σ^*.

Next we recall the description of the H-fixed elements in the distribution module $(\mathcal{H}_{\sigma,\lambda}^\infty)^*$. We first set for each $w \in \mathcal{W}_H\backslash\mathcal{W}$

$$V^*(\sigma, w) := (V_\sigma^*)^{w^{-1}M_H w} \,.$$

Note that this space is one-dimensional. Set

$$V^*(\sigma) := \bigoplus_{w \in \mathcal{W}_H\backslash\mathcal{W}} V^*(\sigma, w) \simeq \mathbb{C}^{|\mathcal{W}_H\backslash\mathcal{W}|}$$

and for $w \in \mathcal{W}_H\backslash\mathcal{W}$ we denote by

$$V^*(\sigma) \to V^*(\sigma, w), \quad \eta \mapsto \eta_w \,.$$

the orthogonal projection. In the sequel we will use the terminology $\operatorname{Re}\lambda >> 0$ if
$$(\operatorname{Re}\lambda - \rho)(\alpha^\vee) > 0 \quad \forall \alpha \in \Sigma^+ .$$
Then, for $\operatorname{Re}\lambda >> 0$ the description

$$j(\sigma^*, -\lambda)(\eta)(g) = \begin{cases} a^{-\rho+\lambda}\sigma^*(m^{-1})\eta_w & \text{if } g = hwman \in HwMAN\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

defines a continuous H-fixed element in $\mathcal{H}_{\sigma^*,-\lambda}$. We may meromorphically continue $j(\sigma^*, \cdot)$ in the λ-variable and obtain, for generic values of λ the identity
$$j(\sigma^*, -\lambda)(V^*(\sigma)) = ((\mathcal{H}^\infty_{\sigma,\lambda})^*)^H .$$
For large λ the inverse map to j is given by
$$((\mathcal{H}^\infty_{\sigma,\lambda})^*)^H \ni \nu \mapsto (\nu(w))_{w\in\mathcal{W}_H\backslash\mathcal{W}} \in V^*(\sigma)\,.$$

For a smooth vector $v \in \mathcal{H}_{\sigma,\lambda}$ and $\eta \in V(\sigma^*)$ we obtain a smooth function on $Y = G/H$ by setting

$$F_{v,\eta}(gH) = \langle \pi_{\sigma,\lambda}(g^{-1})v, j(\sigma^*, -\lambda)(\eta)\rangle\,.$$
The Plancherel theorem for $L^2(Y)_{\mathrm{mc}}$, see for instance [1], then asserts the existence of a meromorphic assignment

$$\mathfrak{a}^*_{\mathbb{C}} \to \mathrm{Gl}(V(\sigma^*)), \quad \lambda \mapsto C(\sigma, \lambda)$$
such that with $j^0(\sigma, \lambda) := j(\sigma, \lambda) \circ C(\sigma, \lambda)$ the map

$$\Phi : \widehat{\bigoplus}_{\sigma\in\widehat{M/M_H}} \int^{\oplus}_{i\mathfrak{a}^*_+} \mathcal{H}_{\sigma,\lambda} \otimes V^*(\sigma)\ d\lambda \to L^2(Y)_{\mathrm{mc}}$$
which for smooth vectors on the left is defined by
$$\sum_\sigma (v_{\sigma,\lambda} \otimes \eta)_\lambda \mapsto \left(gH \mapsto \sum_\sigma \int_{i\mathfrak{a}^*_+} F_{v_{\sigma,\lambda}, j^0(\sigma^*,-\lambda)(\eta)}(gH)\ d\lambda \right)$$
extends to a unitary G-equivalence. Here $\mathfrak{a}^*_+$ denotes a Weyl chamber in $\mathfrak{a}^*$.

Remark 5.2. *Suppose that $\mathcal{W} = \mathcal{W}_H$ (this happens in the group case). Then $V(\sigma^*)$ is one dimensional and we obtain with Remark 5.1 the following isomorphism:*

$$\widehat{\bigoplus}_{\sigma\in\widehat{M/M_H}} \int^{\oplus}_{i\mathfrak{a}^*_+} \mathcal{H}_{\sigma,\lambda} \otimes V(\sigma^*)\ d\lambda \simeq \int_{i\mathfrak{a}^*_+} L^2(\Xi)_\lambda\ d\lambda\,.$$
Hence we may view Φ as defined on a subspace of $L^2(\Xi)$.

The inverse of the map Φ is the most continuous Fourier transform $\mathcal{F}$ (or $\mathcal{F}_{mc}$). For $f \in L^2(Y)_{\mathrm{mc}} \cap L^1(Y)$ the Fourier-transform is given by

$$\mathcal{F}(f)(\sigma, \lambda, \eta)(g) := \int_Y f(y) j^0(\sigma, \lambda)(\eta)(y^{-1}g)\ dy\,,$$

where $\sigma \in \widehat{M/M_H}$, $\lambda \in i\mathfrak{a}^*$ and $\eta \in (V^*(\sigma))^* \simeq V(\sigma^*)$. As a last piece of information we need to relate the Fourier-transform and the Radon-transform.

5.1. **The relation between Fourier and Radon transform**

Now we can determine the relation between $\mathcal{R}$ and $\mathcal{F}$. Let us write $\mathcal{F}_A^w$ for $\mathcal{F}_A$ on $\Xi_w = G/M_H N^w$. Let $f \in C_c^\infty(Y)$.

We unwind definitions:

$$\begin{aligned}
&\mathcal{F}(f)(\sigma, \lambda, \eta)(g) = \int_Y f(gy) j^0(\sigma, \lambda)(\eta)(y^{-1})\ dy \\
&= \sum_{w \in \mathcal{W}/\mathcal{W}_H} \int_{ANM/wM_Hw^{-1}} f(ganmw \cdot y_o) \\
&\quad \cdot j^0(\sigma, \lambda)(\eta)(w^{-1}m^{-1}a^{-1}n^{-1})\ da\ dn\ dm \\
&= \sum_{w \in \mathcal{W}/\mathcal{W}_H} \int_{AM/wM_Hw^{-1}} \mathcal{R}_w(f)(gmawM_HN^w) \\
&\quad \cdot a^{\rho+\lambda} j^0(\sigma, \lambda)(w^{-1}m^{-1})\ da\ \ dm \\
&= \sum_{w \in \mathcal{W}/\mathcal{W}_H} \int_{M/wM_Hw^{-1}} [\mathcal{F}_A^w \circ \mathcal{R}_w](f)(w^{-1}\lambda, gmwM_HN^w) \\
&\quad \cdot j^0(\sigma, \lambda)(w^{-1}m^{-1})\ dm \\
&= \sum_{w \in \mathcal{W}/\mathcal{W}_H} \int_{M/wM_Hw^{-1}} [\mathcal{F}_A^w \circ \mathcal{R}_w](f)(w^{-1}\lambda, gmwM_HN^w) \\
&\quad \cdot \sigma(m) j^0(\sigma, \lambda)(\eta)(w^{-1})\ \ dm
\end{aligned}$$

Let us remark that $j^0(\sigma, \lambda)(\eta)$ is a distribution and a priori the evaluation $j^0(\sigma, \lambda)(\eta)(w^{-1})$ has only meaning for $\operatorname{Re}\lambda$ sufficiently small. This problem is overcome by the meromorphic continuation of $j(\sigma, \lambda)$. This meromorphic continuation is in fact obtainable by an iterative procedure starting with $\operatorname{Re}\lambda$ small and larger values obtained by a differential operator with polynomial coefficients [4]. This fact allows

us to replace $C_c^\infty(Y)$ with the Schwartz space $\mathcal{S}(Y)$ of rapidly decreasing functions (see [2] Sect. 12, and not be confused with the Harish-Chandra Schwartz space $\mathcal{C}(Y)$ in Section 5 below). Thus we have shown:

Lemma 5.3. *Let $f \in \mathcal{S}(Y)$. Then for all $\sigma \in \widehat{M/M_H}$, $\lambda \in i\mathfrak{a}_+^*$*

$$\mathcal{F}(f)(\sigma,\lambda,\eta)(g) = \sum_{w\in\mathcal{W}/\mathcal{W}_H} \int_{M/wM_Hw^{-1}} [\mathcal{F}_A^w \circ \mathcal{R}_w](f)(w^{-1}\lambda, gmwM_HN^w)$$
$$\cdot\, \sigma(m)j^0(\sigma,\lambda)(\eta)(w^{-1})\ dm\,.$$

Remark 5.4. *The special case of $\mathcal{W} = \mathcal{W}_H$ is of particular interest. Then the formula from above simplifies to*

$$\mathcal{F}(f)(\sigma,\lambda,\eta)(g) = \int_{M/M_H} [\mathcal{F}_A \circ \mathcal{R}](f)(\lambda, gmM_HN)$$
$$\cdot\, \sigma(m)j^0(\sigma,\lambda)(\eta)(\mathbf{1})\ dm\,.$$

Theorem 5.5. *$\mathcal{R}$ restricted to $\mathcal{A}_{\rm mc}\cap\mathcal{S}(Y)$ is injective.*

Proof. Let $f \in \mathcal{A}_{\rm mc}\cap\mathcal{S}(Y)$. Suppose that $\mathcal{R}(f) = 0$. With Remark 3.3 we conclude that $\mathcal{R}_w(f) = 0$ for all w. Hence the lemma from above implies that $\mathcal{F}(f) = 0$. As the Fourier transform is injective on $\mathcal{S}(Y)$, see [2] Cor. 12.7, we get that $f = 0$. □

Remark 5.6. *It is very likely that $\mathcal{S}(Y)\cap\mathcal{A}_{\rm mc}$ is dense in $\mathcal{A}_{\rm mc}$, but there does not exist a reference at the moment. If this were established, then the theorem above would imply that $\mathcal{R}$ restricted to $\mathcal{A}_{\rm mc}$ is injective.*

5.2. **Concluding remarks**

5.2.1. *The group case.* It is instructive to see what the results in this paper mean for a semisimple group G viewed as a symmetric space, i.e.:

$$G \simeq G\times G/\Delta(G)$$

with $\Delta(G) = \{(g,g) \mid g\in G\}$ the diagonal group. If $P = MAN$ is a minimal parabolic of G and $\overline{P} = MA\overline{N}$ its standard opposite (i.e. the image under the corresponding Cartan involution), then the parameter space for the horospheres is given by

$$\Xi = G\times G/\Delta(MA)(N\times\overline{N})\,.$$

Our function space $\mathcal{A}$ are then the analytic vectors for the left-right regular representation of $G\times G$ on $L^1(G)$. For $f\in\mathcal{A}$ one then has

$$\mathcal{R}(f)((g,h)\Delta(MA)(N\times\overline{N})) = \int_{N\times\overline{N}} f(gn\overline{n}h^{-1})\; dn\; d\overline{n}\,.$$

5.2.2. *The next steps.* Coming back to our more general situation of $Y=G/H$ let us consider the double fibration

$$\begin{tikzcd} & G/M_H \arrow[dl] \arrow[dr] & \\ \Xi & & Y\,. \end{tikzcd} \tag{5.1}$$

With $\mathcal{R}$ comes a dual transform $\mathcal{R}^\vee$ between appropriate function spaces $\mathcal{F}(\Xi)$ and $\mathcal{F}(Y)$ on Ξ and Y:

$$\mathcal{F}(\Xi)\to\mathcal{F}(Y);\ \mathcal{R}^\vee(\phi)(gH) = \int_{H/M_H} \phi(gh\cdot M_H N)\; d(hM_H)\,.$$

For $f\in\mathcal{A}_{\mathrm{mc}}$ one then might ask about the existence of a pseudo-differential operator D such that

$$f=\mathcal{R}^\vee(D\mathcal{R}(f))$$

holds. For $Y=\mathrm{Sl}(2,\mathbb{R})/\mathrm{SO}(1,1)$ this was considered in [7] where it was shown that such a pseudo-differential operator D exists. For Y being a group one might expect that D is in fact a differential operator.

5.2.3. *Radon transform on Schwartz spaces.* One might ask to what extend $\mathcal{R}$ might be defined on the Schwartz space of Y. For some classes of Y this seems to be possible and we will comment on this in more detail below. Let us first recall the definition of the Schwartz space.

One uses

$$G=KAH \tag{5.2}$$

often referred to as the polar decomposition of G (with respect to H and K). Accordingly every $g\in G$ can be written as $g=k_g a_g h_g$ with $k_g\in K$ etc. It is important to notice that a_g is unique modulo $\mathcal{W}_H$. Therefore the prescription

$$\|gH\| := |\log a_g| \qquad (g\in G)$$

is well defined for $|\cdot|$ the Killing norm on $\mathfrak{p}$. An alternative, and often useful, description of $\|\cdot\|$ is as follows

$$\|y\| = \frac{1}{4}\left|\log\left[y\tau(y)^{-1}\theta(y\tau(y)^{-1})^{-1}\right]\right| \qquad (y\in Y)\,. \tag{5.3}$$

For $u \in \mathcal{U}(\mathfrak{g})$ we write L_u for the corresponding differential operator on Y, i.e. for $u \in \mathfrak{g}$

$$(L_u f)(y) = \frac{d}{dt}\Big|_{t=0} f(\exp(-tu)y)\,,$$

whenever f is a differentiable function at y. With these preliminaries one defines the Harish-Chandra Schwartz space of Y by

$$\mathcal{C}(Y) = \{f \in C^\infty(Y) \mid \forall u \in \mathcal{U}(\mathfrak{g})\ \forall n \in \mathbb{N}$$
$$\sup_{y \in Y} \Theta(y)(1 + \|y\|)^n |(L_u f)(y)| < \infty\}$$

where $\Theta(gH) = \phi_0(g\tau(g)^{-1})^{-1/2}$ and ϕ_0 Harish-Chandra's basic spherical function.

It is not to hard to see that $\mathcal{C}(Y)$ with the obvious family of defining seminorms is a Fréchet space. Moreover $\mathcal{C}(Y)$ is G-invariant and G acts smoothly on it. We note that $\mathcal{C}(Y) \subset L^2(Y)$ is a dense subspace.

We write $BC^\infty(\Xi)$ for the space of bounded smooth functions on Ξ.

In the context of defining $\mathcal{R}$ on $\mathcal{C}(Y)$ we focus we wish to discuss a basic example.

Lemma 5.7. *Let* $Y = \mathrm{Sl}(2,\mathbb{R})/\mathrm{SO}(1,1)$*, and* $f \in \mathcal{C}(Y)$*. Then the following assertions hold:*

(i) *The integral* $\int_N f(nH)\ dn$ *is absolutely convergent.*

(ii) *The prescription*

$$gM_H N \mapsto \int_N f(gnH)\ dn$$

defines a function in $BC^\infty(\Xi)$.

Proof. Let A be the diagonal subgroup of G (with positive entries) and $N = \begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix}$.

(i) For $x \in \mathbb{R}$ and $n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ we have to determine $a_x \in A$ such that $n_x \in K a_x H$. We use (5.3) and start:

$$z_x := n_x \tau(n_x)^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} = \begin{pmatrix} 1-x^2 & -x \\ x & 1 \end{pmatrix}$$

and hence

$$\begin{aligned} y_x := z_x \theta(z_x)^{-1} &= \begin{pmatrix} 1-x^2 & -x \\ x & 1 \end{pmatrix} \cdot \begin{pmatrix} 1-x^2 & x \\ -x & 1 \end{pmatrix} \\ &= \begin{pmatrix} (1-x^2)^2 + x^2 & * \\ * & 1+x^2 \end{pmatrix}. \end{aligned}$$

For $|x|$ large we have $\log|y_x| = |\log y_x|$. Furthermore up to an irrelevant constant

$$\begin{aligned}|y_x| = [\operatorname{tr}(y_x y_x)]^{\frac{1}{2}} \geq \frac{1}{2}[(1-x^2)^2 + x^2 + 1 + x^2] \\ \geq \frac{1}{2}[x^4+1]\end{aligned}$$

Therefore, for $|x|$ large

$$\|n_x\| \geq \frac{1}{4}\log(x^4/2 + 1/2)$$

From Harish-Chandra's basic estimates of ϕ_0 and our computation of z_x we further get that

$$\Theta(n_x) \geq |x|\,.$$

Therefore for $f \in \mathcal{C}(Y)$ we obtain that $x \mapsto |f(n_x H)|$ growths slower than $\frac{1}{|x|\cdot|\log x|^N}$ for any fixed $N > 0$ and $|x|$ large. This shows (i).
(ii) Let $f \in \mathcal{C}(Y)$ and set $F := \mathcal{R}(f)$. From the proof of (i) we know that F is smooth. It remains to see that F is bounded. From $G = KAH$ we deduce that it is enough to show that $F|_A$ is bounded. We do this by direct computation. For $t > 0$ we set

$$a_t = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}.$$

Then

$$a_t n_x = \begin{pmatrix} t & tx \\ 0 & 1/t \end{pmatrix}$$

and thus

$$\begin{aligned} z_{t,x} := a_t n_x \tau(a_t n_x)^{-1} &= \begin{pmatrix} t & tx \\ 0 & 1/t \end{pmatrix} \cdot \begin{pmatrix} t & 0 \\ -tx & 1/t \end{pmatrix} \\ &= \begin{pmatrix} t^2(1-x^2) & -x \\ x & 1/t^2 \end{pmatrix}. \end{aligned}$$

With that we get

$$y_{t,x} = z_{t,x}\theta(z_{t,x})^{-1} = \begin{pmatrix} t^4(1-x^2)^2 + x^2 & * \\ * & 1/t^4 + x^2 \end{pmatrix}.$$

For $t \geq 1$ we conclude that

$$\|a_t n_x\| \gtrsim \log\left(\begin{cases} c_1 t^4 & \text{for } |x| \leq 1/2, \\ c_2 t^4 x^4 - c_3 & \text{for } |x| \geq 1/2\,. \end{cases}\right)$$

and for $|t| < 1$ one has

$$\|a_t n_x\| \geq \log |x| .$$

From that we obtain (ii). $\square$

This example is somewhat specific. One might expect that the Radon transform on $\mathcal{C}(Y)$ converges whenever the real rank of G and of Y coincide.

For groups it is not hard to show that $\mathcal{R}(f)$ does not converge for general $f \in \mathcal{C}(G)$; integrability of f is needed.

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